

Photon polarization tensor in a magnetized plasma system

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We investigate the photon polarization tensor at finite temperature in the presence of a static and homogeneous external magnetic field. In our scheme, the Matsubara frequency summation is performed after Poisson summation, which will be taken easily and convergent quickly in the frame of proper time representation. Moreover, the dependence of Landau levels is expressed explicitly. It demonstrates the convergence of summing Landau levels as it has to be. Consequently, there is no necessary to truncate the Landau level in a numerical estimation. At zero temperature, the Lowest Landau Level (LLL) approximation is analytically satisfied for the imaginary parts of the vacuum photon polarization tensor. Our results examine that, the LLL approximation is not enough for the thermal photon polarization tensor, it gains the contribution not only from the lowest Landau level but also up to the finite- n levels. Such large imaginary ones only show up at finite temperatures, which is the so called Landau damping. It originates from the absorption of soft fields by hard plasma constituents, which is a universal feature of plasma systems. Finally, it was argued that the summation of Matsubara frequency is not commuted with Landau level ones, such conjecture is excluded in our calculations.

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I. INTRODUCTION

At the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC), strong magnetic fields with the magnitude of $10^{18} \sim 10^{20}$ G [corresponding to $eB \sim (0.1 - 1.0 \text{ GeV})^2$], can be generated through non-central heavy-ion collisions [1, 2]. In recent years, studies of Quantum Chromodynamics (QCD) phase structure and phase diagram under strong external magnetic fields attracted lots of interests [3–5], and it is found that QCD matter under strong magnetic fields exhibits some novel properties, e.g. the chiral magnetic effect (CME) [6–8], the magnetic catalysis (MC) [9–11] and inverse magnetic catalysis (IMC) [12–14], a possible formation of the vacuum superconductor [15, 16] and its survival at high temperature [17, 18], etc. With such strong magnetic fields, Quantum Electrodynamics (QED) will also be modified, such as photon decay into an electron-positron pair via Schwinger mechanism [19–21], vacuum birefringence of a photon [22, 23], photon splitting and so on [24, 25].

The strong magnetic field is generated in the early stage of non-central heavy-ion collisions, and then fast decays. It is intriguing to know whether and how properties of the created quark-gluon-plasma (QGP) will be modified by the strong magnetic field, and whether it can be measured in experiment. Dileptons and photons are produced through the bulk of the matter and throughout the entire history of the collision, thus can be regarded as “penetrating probes” of QGP. Recently, experimental evidences of photon anisotropy from the PHENIX Collaboration [26] challenges existing theoretical models, and it was proposed in [27] that the photon anisotropy might be induced by a large anisotropic magnetic field in heavy ion collisions. This raises interests of study the photon and dilepton production in the presence of an external magnetic field [28, 29].

The photon polarization tensor carries the fundamental information of magnetized vacuum or medium [30–34]. A complete description of the vacuum polarization tensor is particularly complicated to approach, since the vacuum photon polarization tensor is expressed in terms of a double summation of infinite series with respect to two Landau levels occupied by virtual charged particles. Most works were focusing on the strong field limit with an assumption of Lowest Landau Level (LLL) [31, 35]. In [36] we obtained a full description of vacuum polarization tensor in response to all the Landau levels at any field strength of B for the first time beyond LLL approximation, and we found out that the imaginary part of the photon polarization tensor $\text{Im} \Pi$ becomes nonzero at the time like momenta region $Q^2 > 4(M^2 + 2neB)$ at $T = 0$, i.e., the LLL approximation is analytically satisfied [32, 36].

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It was not fully understood of the imaginary parts of thermal photon polarization tensor in a magnetized media. The main purpose of this paper is to investigate whether the above conclusion will be influenced by temperatures, which, in turn, is not keeping suitable. At finite temperature, in our scheme, the Matsubara frequency was summed after applying Poisson summation formula, which finished easily and convergent quickly under the help of proper time representation. It was argued that the summation of Matsubara frequency is not commuted with Landau level ones [37]. Our calculation exclude this conjecture. The summation of Landau level is convergent as it has to be since no more divergent environment included. However, in the early works, one has to test by a numerical way to find out the cutoff of the Landau level. In our work, the dependence of Landau level is expressed in an obviously matter. Therefore, one is able to truncate the Landau level via a systematic consideration while proceeding numerical simulation.

The paper is organized as follows. We show the vacuum photon polarization tensor $\Pi^{\mu\nu}$ at external B in section II. In section III, we present the magnetized photon polarization tensor $\Pi^{\mu\nu}$ at finite temperature. Due to the additional vector of the velocity of the plasma, the decomposition of the photon polarization tensor will become more complicated. The full expression will be examined in section IV. We discuss the limiting behaviors of thermal photon polarization tensor $\Pi^{\mu\nu}$ and physical explanations at different kinematics regimes in section V. We end up with the summary and future applications in section VI.

II. PHOTON POLARIZATION TENSOR IN THE VACUUM

Following Ref. [35], the decomposed fermion propagator $\mathcal{S}(k)$ in a static and homogeneous external magnetic field can be written as :

$$\mathcal{S}(k) = i \exp\left(2\hat{k}_\perp^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \mathcal{D}_n(eB, k)}{k_0^2 - k_3^2 - M^2 - 2neB}, \quad (1)$$

with

$$\begin{aligned} \mathcal{D}_n(eB, k) &= 2(\not{k}_\parallel + M) \mathcal{O}^- L_n(-4\hat{k}_\perp^2) \\ &\quad - 2(\not{k}_\parallel + M) \mathcal{O}^+ L_{n-1}(-4\hat{k}_\perp^2) - 4\not{k}_\perp L_{n-1}^1(-4\hat{k}_\perp^2). \end{aligned} \quad (2)$$

Where $\hat{k}_\perp^2 = k_\perp^2 / (2eB)$. $L_n^a(\xi)$ are the generalized Laguerre polynomials, with $L_{n-i}^a(\xi) = 0$ if $n < i$. $\mathcal{O}^\pm = (1 \pm i\gamma^1\gamma^2)/2$ are the projecting operators corresponding to the spin state of charged particle paralleling or anti-paralleling to the direction of external field B , and they satisfy following commutation relations:

$$\begin{aligned} \mathcal{O}^\pm \gamma^\mu \mathcal{O}^\pm &= \mathcal{O}^\pm \gamma_\parallel \\ \mathcal{O}^\pm \gamma^\mu \mathcal{O}^\mp &= \mathcal{O}^\pm \gamma_\perp. \end{aligned} \quad (3)$$

Here, the metric convention $g^{\mu\nu}$ is decomposed into two orthogonal subspaces

$$g_\parallel^{\mu\nu} = \text{diag}(1, 0, 0, -1), \quad g_\perp^{\mu\nu} = \text{diag}(0, -1, -1, 0). \quad (4)$$

Similar decompositions are adopted for four dimensional momentum $k^\mu = k_\parallel^\mu + k_\perp^\mu$ with

$$k_\parallel^\mu = (k^0, 0, 0, k^3), \quad k_\perp^\mu = (0, k^1, k^2, 0) \quad (5)$$

and Dirac matrices

$$\gamma_\parallel^\mu = (\gamma^0, 0, 0, \gamma^3), \quad \gamma_\perp^\mu = (0, \gamma^1, \gamma^2, 0). \quad (6)$$

In the vacuum, the photon polarization tensor is expressed as

$$\Pi^{\mu\nu}(q) = -ie^2 \text{Tr}[\mathcal{S}(k) \gamma^\mu \mathcal{S}(p) \gamma^\nu], \quad (7)$$

with $p = k + q$. Following the procedure described in [36], the vacuum polarization tensor can be derived as:

$$\Pi^{\mu\nu} = \int d\Gamma \, I^{\mu\nu} e^{-[\hat{M}^2 - \eta \hat{q}_\parallel^2 + nx + m(1-x) - \hat{k}_\parallel^2 - i\epsilon]\tau}, \quad (8)$$

where $\eta = x(1-x)$ and

$$d\Gamma = -ie^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{n+m} \int_0^1 dx \int_0^{\infty} d\tau \tau \cdot \int \frac{d^{2-\epsilon} \hat{k}_{||}}{(2\pi)^2} \int \frac{d^2 \hat{k}_{\perp}}{(2\pi)^2} \exp \left[2\hat{k}_{\perp}^2 + 2\hat{p}_{\perp}^2 \right]. \quad (9)$$

and the tensor structure of vacuum polarization $I^{\mu\nu}$ becomes

$$\begin{aligned} I^{\mu\nu} = & 2\text{Tr} \left[\not{k}_{||}^{1-x} \gamma_{||}^{\mu} \not{k}_{||}^x \gamma^{\nu} \right] (L_n L_m + L_{n-1} L_{m-1}) \\ & - 2\text{Tr} \left[\not{k}_{||}^{1-x} \gamma_{\perp}^{\mu} \not{k}_{||}^x \gamma^{\nu} \right] (L_n L_{m-1} + L_{n-1} L_m) \\ & - 4\text{Tr} \left[\not{k}_{||}^{1-x} \gamma^{\mu} \not{k}_{\perp}^{\alpha} \gamma^{\nu} \right] (L_n - L_{n-1}) L_{m-1}^1 \\ & - 4\text{Tr} \left[\not{k}_{\perp}^{\beta} \gamma^{\mu} \not{k}_{||}^x \gamma^{\nu} \right] L_{n-1}^1 (L_m - L_{m-1}) \\ & + 16\text{Tr} \left[\not{k}_{\perp}^{\beta} \gamma^{\mu} \not{k}_{\perp}^{\alpha} \gamma^{\nu} \right] L_{n-1}^1 L_{m-1}^1, \end{aligned} \quad (10)$$

where

$$\begin{aligned} \not{k}_{||}^x &= \not{k}_{||} + x \not{q}_{||} + M, \quad \not{k}_{||}^{1-x} = \not{k}_{||} - (1-x) \not{q}_{||} + M, \\ \not{k}_{\perp}^{\alpha} &= \not{k}_{\perp} + \frac{\alpha}{\alpha + \beta} \not{q}_{\perp}, \quad \not{k}_{\perp}^{\beta} = \not{k}_{\perp} - \frac{\beta}{\alpha + \beta} \not{q}_{\perp}. \end{aligned} \quad (11)$$

The arguments of $L_{n,m}^{\alpha,\beta}$ have been abbreviated.

III. PHOTON POLARIZATION TENSOR AT FINITE TEMPERATURE

At finite temperature, the temperature T breaks Lorentz invariance, and we have to separate the zeroth component of the momentum. To calculate $\Pi^{\mu\nu}$, a general Feynman parameter, x , for the denominator factor has been introduced [38]. Besides, the proper time representation of fermion propagator is employed [39]. Then, the denominator factor is shown as:

$$\frac{1}{ab} = \int_0^1 dx \int_0^{\infty} d\tau \tau \exp [(xa + (1-x)b) \tau] \quad (12)$$

where $a = k_{||}^2 - M^2 - 2neB$ and $b = (k + q)_{||}^2 - M^2 - 2meB$ in our work. The longitudinal momentum is denoted as $k_{||}^2 = k_0^2 - k_3^2$, while the transverse momentum is $k_{\perp}^2 = -k_1^2 - k_2^2$. τ is the variable of proper time. In this paper, we normalize all the energy scale to dimensionless, where $\hat{q}^2 = q^2/(2eB)$, $\hat{M}^2 = M^2/(2eB)$, and so on.

It is well known to shift k to $k - (1-x)q$ can simplify the calculations. But, as pointed out before, because the external magnetic field breaks the uniform of the $3-d$ space, it is only allowed to shift k_3 to $k_3 - (1-x)q_3$ along the direction of the B -field as usual. And the transverse momentum k_{\perp} has to be shifted $k_{\perp} - q_{\perp}\beta/(\alpha + \beta)$ shown later by the explicit expression of Eq. (18). The detail notations α, β are described below. We remark that there is no shifting on the zero component since the integration with respect to k_0 is replaced by summing over Matsubara frequencies $2i\pi T \sum_l$. Hence, we have

$$\Pi^{\mu\nu} = \int d\Gamma I^{\mu\nu} \exp \left[- \left(\hat{M}^2 - \eta \hat{q}_{||}^2 + nx + m(1-x) - \left(\hat{k}_0 + (1-x)\hat{q}_0 \right)^2 + \hat{k}_3^2 \right) \tau \right], \quad (13)$$

where $\eta = x(1-x)$ and the volume space is

$$d\Gamma = -ie^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{n+m} \int_0^1 dx \int_0^{\infty} d\tau \int \frac{d^2 \hat{k}_{||}}{(2\pi)^2} \int \frac{d^2 \hat{k}_{\perp}}{(2\pi)^2} \tau \exp \left[2\hat{k}_{\perp}^2 + 2\hat{p}_{\perp}^2 \right]. \quad (14)$$

The tensor structure of thermal polarization $I^{\mu\nu}$ is expressed as:

$$\begin{aligned}
I^{\mu\nu} = & 2\text{Tr} \left[\not{k}_{||}^{1-x} \gamma_{||}^{\mu} \not{k}_{||}^x \gamma^{\nu} \right] (L_n L_m + L_{n-1} L_{m-1}) - 2\text{Tr} \left[\not{k}_{||}^{1-x} \gamma_{\perp}^{\mu} \not{k}_{||}^x \gamma^{\nu} \right] (L_n L_{m-1} + L_{n-1} L_m) \\
& - 4\text{Tr} \left[\not{k}_{||}^{1-x} \gamma^{\mu} \not{k}_{\perp}^{\alpha} \gamma^{\nu} \right] (L_n - L_{n-1}) L_{m-1}^1 - 4\text{Tr} \left[\not{k}_{\perp}^{\beta} \gamma^{\mu} \not{k}_{||}^x \gamma^{\nu} \right] L_{n-1}^1 (L_m - L_{m-1}) \\
& + 16\text{Tr} \left[\not{k}_{\perp}^{\beta} \gamma^{\mu} \not{k}_{\perp}^{\alpha} \gamma^{\nu} \right] L_{n-1}^1 L_{m-1}^1 + 2i\text{Tr} \left[\not{k}_{||}^{1-x} \gamma^1 \gamma^2 \gamma_{\perp}^{\mu} \not{k}_{||}^x \gamma^{\nu} \right] (L_n L_{m-1} - L_{n-1} L_m) \\
& + 4i\text{Tr} \left[\not{k}_{||}^{1-x} \gamma^1 \gamma^2 \gamma^{\mu} \not{k}_{\perp}^{\alpha} \gamma^{\nu} \right] (L_n + L_{n-1}) L_{m-1}^1 + 4i\text{Tr} \left[\not{k}_{\perp}^{\beta} \gamma^{\mu} \not{k}_{||}^x \gamma^1 \gamma^2 \gamma^{\nu} \right] L_{n-1}^1 (L_m + L_{m-1}), \quad (15)
\end{aligned}$$

where

$$\begin{aligned}
\not{k}_{||}^x &= (k_0 + q_0) \gamma^0 - (k_3 + x q_3) \gamma^3 + M, \\
\not{k}_{||}^{1-x} &= k_0 \gamma^0 - (k_3 + x q_3 - q_3) \gamma^3 + M, \\
\not{k}_{\perp}^{\alpha} &= \not{k}_{\perp} + \not{q}_{\perp} \alpha / (\alpha + \beta), \\
\not{k}_{\perp}^{\beta} &= \not{k}_{\perp} - \not{q}_{\perp} \beta / (\alpha + \beta). \quad (16)
\end{aligned}$$

The arguments of $L_{n,m}^{\alpha,\beta}$ have been abbreviated. Under the help of generating function of Laguerre polynomials [40]:

$$\sum_{n=0}^{\infty} t^n L_{n-i}^a(\xi) = \frac{t^i}{(1-t)^{a+1}} \exp \left[\frac{-t\xi}{1-t} \right] \quad (17)$$

for $|t| < 1$, we are able to evaluate the summation of Landau level in a direct manner. We have

$$\begin{aligned}
& \exp \left[2\hat{k}_{\perp}^2 + 2\hat{p}_{\perp}^2 \right] \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{(n+m)} \exp \left[-(nx + m(1-x))\tau \right] L_{n-i}^a(-4\hat{k}_{\perp}^2) L_{m-j}^b(-4\hat{p}_{\perp}^2) \\
&= \frac{t_1^i t_2^j}{(1-t_1)^{a+1} (1-t_2)^{b+1}} \exp \left[\frac{\alpha\beta}{\alpha+\beta} \hat{q}_{\perp}^2 \right] \exp \left[(\alpha+\beta) \left(\hat{k}_{\perp} + \frac{\beta}{\alpha+\beta} \hat{q}_{\perp} \right)^2 \right]. \quad (18)
\end{aligned}$$

The last exponential term explains the unusual shifting of transverse momentum which is early taken in Eq. (15). Here, $t_1 = -e^{-x\tau}$, $t_2 = -e^{-(1-x)\tau}$, $\alpha = 2(1+t_1)/(1-t_1)$ and $\beta = 2(1+t_2)/(1-t_2)$.

IV. TENSOR STRUCTURES AT FINITE TEMPERATURE

In a thermal plasma, we introduce the velocity of the fluid u . It combines with the four momentum of particles to create three second order tensor $u \otimes u$, $u \otimes q$ and $q \otimes q$. Including the metric $g^{\mu\nu}$, the electromagnetic tensor $F_{\mu\nu}$ and dual tensor $\tilde{F}_{\mu\nu} = \epsilon_{\alpha\beta\mu\nu} F^{\alpha\beta}$, we have six independent second order tensors. Because of the requirement of Wald identity $\Pi^{\mu\nu} q_{\nu} = 0$ and the without loss of generality choice $u^{\mu} u_{\mu} = 1$, it means we are able to separate $\Pi^{\mu\nu}$ into four independent structures. From the definition $\Pi^{\mu\nu} = -ie^2 \text{Tr} [\not{\mathcal{S}}(k + q/2) \gamma^{\mu} \not{\mathcal{S}}(k - q/2) \gamma^{\nu}] = -ie^2 \text{Tr} [\not{\mathcal{S}}(k - q/2) \gamma^{\nu} \not{\mathcal{S}}(k + q/2) \gamma^{\mu}]$, it draws a conclusion that $\Pi^{\mu\nu}(q) = \Pi^{\nu\mu}(-q)$ [24, 25]. Therefore, $\Pi^{\mu\nu}$ contains symmetry parts made by even powers of four momentum q and antisymmetric parts which formed by odd powers of q . To find out the subspace, we set up four mutual orthogonal four momentums:

$$x_0 = q^{\mu}; \quad x_1 = \tilde{F}^{\mu\rho} q_{\rho}; \quad x_2 = F^{\mu\rho} q_{\rho}; \quad x_3 = u^{\mu} - x_0^{\mu} \frac{u \cdot x_0}{x_0^2} - x_1^{\mu} \frac{u \cdot x_1}{x_1^2} - x_2^{\mu} \frac{u \cdot x_2}{x_2^2}, \quad (19)$$

where the fluid velocity $u = (1, 0, 0, 0)$. Hence, the associated transversed symmetric tensors are

$$P_1^{\mu\nu} = \frac{x_1^{\mu} x_1^{\nu}}{x_1^2}; \quad P_2^{\mu\nu} = \frac{x_2^{\mu} x_2^{\nu}}{x_2^2}; \quad P_3^{\mu\nu} = \frac{x_3^{\mu} x_3^{\nu}}{x_3^2}, \quad (20)$$

which satisfy the following relationship:

$$P_i^{\mu\nu} = P_i^{\nu\mu}; \quad P_i^{\mu\nu} q_{\nu} = 0; \quad P_i^2 = P_i; \quad P_i P_j = 0; \quad \sum_{i=1}^3 P_i^{\mu\nu} = g^{\mu\nu} - \frac{q^{\mu} q^{\nu}}{q^2}, \quad (21)$$

with $i = 1, 2, 3$ and $i \neq j$. Under the help of the fluid velocity u , the antisymmetric tensor must contain the form structures of $u^\mu x_i^\nu - x_i^\nu u^\mu$ for $i = 0, 1, 2$; the natural candidates $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$. In fact, we obtain two independent structure forms, which satisfy Wald identity as required. But, one is totally null in our simple electromagnetic tensor environment. Another one is $P_A^{\mu\nu}(q) = P_A^{\nu\mu}(-q) = u^\mu x_2^\nu - x_2^\mu u^\nu + (u \cdot q) F^{\mu\nu}$. We normalize the conjugate dimensionless projector

$$P_4^{\mu\nu} = (P_4^{\mu\nu})^\dagger = \frac{i u^\mu x_2^\nu - i x_2^\mu u^\nu}{u \cdot q} + i F^{\mu\nu}. \quad (22)$$

which satisfy $P_4 P_i = 0$ for $i = 1, 2, 3$.

Then, after summing over Landau levels, $\Pi^{\mu\nu}$ takes the form of

$$\Pi^{\mu\nu} = \sum_{i=1}^4 P_i^{\mu\nu} \pi_i = \sum_{i=1}^4 P_i^{\mu\nu} \int d\Gamma_E \frac{4\tau e^{-v\tau}}{(1-t_1)(1-t_2)} \exp[h(x, \tau) \hat{q}_\perp^2] \mathcal{I}_i, \quad (23)$$

where $v = \hat{M}^2 - \eta \hat{q}_\parallel^2$ and $h(x, \tau) = \alpha\beta/(\alpha + \beta)$. The scalar functions π_i are developed by the inner product of $\langle P_i^{\mu\nu} : \Pi^{\mu\nu} \rangle$, which are in the form of as below

$$\begin{aligned} \mathcal{I}_1 &= 2(M^2 + \Theta_1^2)(1 + t_1 t_2) + \frac{4[\alpha\beta q_\perp^2 - k_\perp^2(\alpha + \beta)^2] t_1 t_2}{(\alpha + \beta)(1 - t_1 t_2)}; \\ \mathcal{I}_2 &= 2\Theta_2^2(t_1 + t_2) + \frac{4q_\perp^2 \alpha\beta t_1 t_2}{(\alpha + \beta)(1 - t_1 t_2)}; \\ \mathcal{I}_3 &= \frac{2q_\perp^2}{q^2} (M^2 - \Theta_1^2)(1 + t_1 t_2) + \frac{4q_\perp^2 [\alpha\beta(q_\perp^2 - q_\parallel^2) - k_\perp^2(\alpha + \beta)^2] t_1 t_2}{q^2(\alpha + \beta)(1 - t_1 t_2)} \\ &\quad + \frac{4q_\perp^2}{q^2(1 - t_1 t_2)} \left[(\tilde{k}_0 q_0 - (1-x)q_\parallel^2)(1 - t_1^2)t_2 - (\tilde{k}_0 q_0 + xq_\parallel^2)(1 - t_2^2)t_1 \right] + \frac{2q_\parallel^2 \Theta_2^2}{q^2}(t_1 + t_2); \\ \mathcal{I}_4 &= 4\Theta_2^2(t_1 - t_2) - \frac{4q_\perp^2}{q_0(1 - t_1 t_2)} \left[(\tilde{k}_0 - (1-x)q_0)(1 + t_1)^2 t_2 + (\tilde{k}_0 + xq_0)(1 + t_2)^2 t_1 \right]; \end{aligned} \quad (24)$$

where $\Theta_1^2 = \eta q_\parallel^2 - (q_0^2 + q_3^2) (\tilde{k}_0^2 + k_3^2) / q_\parallel^2 + (1-2x)\tilde{k}_0 q_0$, $\Theta_2^2 = \tilde{k}_0^2 - (1-2x)\tilde{k}_0 q_0 - k_3^2 - M^2 - \eta q_\parallel^2$ and $\tilde{k}_0 = k_0 + (1-x)q_0$. The identity $(1-t_1)(1-t_2)(\alpha + \beta) = 4(1-t_1 t_2)$ has been plugged in. After performing wick rotation, one has $k^2 = -k_E^2$ and the integral volume space is

$$d\Gamma_E = e^2 \int_0^1 dx \int_0^\infty d\tau \int \frac{d\Gamma_{k_\parallel}}{(2\pi)^2} \int \frac{d\Gamma_{k_\perp}}{(2\pi)^2}, \quad (25)$$

where

$$\begin{aligned} \int d\Gamma_{k_\parallel} &= \hat{T} \sum_{l=-\infty}^\infty \exp[-(\hat{\omega}_l - i(1-x)\hat{q}_0)^2 \tau] \int d\hat{k}_3 \exp[-\hat{k}_3^2 \tau], \\ \int d\Gamma_{k_\perp} &= \int d^2 \hat{k}_\perp \exp[-(\alpha + \beta)(\hat{k}_1^2 + \hat{k}_2^2)]. \end{aligned} \quad (26)$$

Note here k_0 has been replaced by $i\omega_l$, where $\hat{\omega}_l^2 = (2l+1)^2 \pi^2 T^2 / (2eB)$.

With respect to the momenta k_\perp^2 , the integrand of $\Pi^{\mu\nu}$ is in a simple Gaussian form. Hence, we are able to write down the polarization tensor by some simple scalar functions in Euclidean space. To simplified, four scalar functions Ξ are denoted as below

$$\Xi(\vartheta, \varrho|v) = \frac{\hat{k}_3^{2\vartheta}}{16\pi^3} \sum_{l=-\infty}^\infty \int_0^\infty d\tau \frac{\hat{T}\tau (i\hat{T})^l (l + \varpi)^l}{1 - e^{-\tau}} \exp\left[-\left(v + \hat{k}_3^2 + (l + \varpi)^2 \hat{T}^2\right) \tau + h(x, \tau) \hat{q}_\perp^2\right]. \quad (27)$$

Here $\hat{T} = (2\pi T)/\sqrt{2eB}$ and $\varpi = \frac{1}{2} - i(1-x)q_0/(2\pi T)$. And then we let $\Xi_i(v) = \Xi(\vartheta = 0, \varrho = i; v)$ for $i = 0, 1, 2$ and $\Xi_3(v) = \Xi(\vartheta = 1, \varrho = 0; v)$. Also, we have some scalar functions are in the form as

$$\Xi_\varrho(v) = \frac{1}{16\pi^3} \sum_{l=-\infty}^\infty \int_0^\infty d\tau \frac{\hat{T}\tau (i\hat{T})^l (l + \varpi)^l}{(1 - e^{-\tau})^2} \exp\left[-\left(v + 1 + \hat{k}_3^2 + (l + \varpi)^2 \hat{T}^2\right) \tau + h(x, \tau) \hat{q}_\perp^2\right] \quad (28)$$

for $\varrho = 0, 1$. Step on, Poisson summation formula have been applied in Eq. (27) and Eq. (28), such as

$$\sum_{l=-\infty}^{\infty} \exp \left[- (l + \varpi)^2 \frac{\pi}{\sigma} \right] = \sum_{l=-\infty}^{\infty} \sigma^{\frac{1}{2}} \exp \left[-l^2 \pi \sigma - 2il\pi\varpi \right], \quad (29)$$

and so on. In this work, we have $\sigma = \pi/(\tau\hat{T}^2)$.

Obviously, the summation over l from $-\infty$ into ∞ is able to split into positive values and zero, i.e. the thermal, Ξ_i^T and vacuum parts, Ξ_i^0 , respectively. All the results of vacuum contribution have been discussed in [36]. In this letter, we focus on the thermal parts.

V. THERMAL RESULTS FOR THE PHOTON POLARIZATION TENSOR

The thermal scalar functions are expressed as below

$$\Xi_i^T(v) = \frac{\hat{k}_3^{2\vartheta}}{16\pi^{\frac{5}{2}}} \sum_{s=\pm 1} \sum_{l=1}^{\infty} \int_0^{\infty} d\tau \frac{\tau^{\frac{1}{2}} s^{\varrho} G(\varrho)}{2\tau(1-e^{-\tau})} \exp \left[- \left(v + \hat{k}_3^2 \right) \tau - \frac{\pi^2 l^2}{\hat{T}^2 \tau} + h(x, \tau) \hat{q}_{\perp}^2 - sl\xi \right] \cos(l\pi), \quad (30)$$

where $\xi = (1-x)q_0/T$. We note here the explicit forms of $G(\varrho) = 2\tau(-\pi l/\hat{T}\tau)^{\varrho} - \Gamma(\varrho+1) + 1$. We rephrase the above equation via the geometirc trick $1/(1-e^{-\tau}) = \sum_{n=0}^{\infty} e^{-n\tau}$. Besides, the thermal parts of Ξ_i^T are easily addressed by $e^{-\tau}/(1-e^{-\tau})^2 = \sum_{n=1}^{\infty} n e^{-n\tau}$ for $i = 0, 1$.

Now, the integral with respect to τ can be written in terms of a modified Bessel function of second kind, $K_{\nu}(y)$, where

$$\int_0^{\infty} \tau^{\nu-1} \exp \left(-\lambda\tau - \frac{\chi}{\tau} \right) d\tau = 2 \left(\frac{\chi}{\lambda} \right)^{\frac{\nu}{2}} K_{\nu} \left(2\sqrt{\lambda\chi} \right), \quad (31)$$

for $\text{Re } \lambda > 0$ and $\text{Re } \chi > 0$. In strong B -field, the most contribution of the integration with respect to τ is from the regime where $\tau \sim (\chi/\lambda)^{\frac{1}{2}} \sim \sqrt{2eB}$. It means we are able to set $h(x, \tau) = 1$ safely. The details reason can be found in the reference [36].

The special case, $\nu = \frac{1}{2}$, of the modified Bessel function at the second kind is $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$. We apply the saddle point method to integrate with respect to x ,

$$\sum_{l=1}^{\infty} \sum_{s=\pm 1} \int_0^1 l^{\frac{1}{2}} y^{-\frac{1}{4}} K_{\frac{1}{2}}(l\sqrt{y}) \exp(sl\xi) \cos(l\pi) dx \simeq \sum_{s=\pm 1} \frac{e^{\frac{i(\pi-\phi)}{2}} \text{Li}_0(-e^{f(x_0)})}{\sqrt{y(x_0)}} \left(\frac{2\pi}{|f^{(2)}(x_0)|} \right)^{\frac{1}{2}} \quad (32)$$

where $f(x) = -\sqrt{y} + s\xi$, $y(x) = (E_k^2 - \eta q_{\parallel}^2)/T^2$ and $E_k^2 = M^2 + 2neB + k_3^2$. x_0 is the saddle point where $f^{(1)}(x_0) = 0$. ϕ is the angle determined by which the contour passes through the saddle point. When $f^{(2)}(x_0) = |r| \pm i\varepsilon$, ϕ is equal to 0 or 2π for positive or negative sign, respectively. To obtain a real root of x_0 locating in the interval $(0, 1)$, it requires $q_{\parallel}^2 \leq 4E_k^2 \leq q_{\parallel}^4/q_0^2$. Therefore, there is no saddle point in the time like momenta for $q_{\parallel}^2 > 0$. Indeed, at the energy region $q_{\parallel}^2 > 4E_k^2$, Eq. (31) is not applicable because λ will be not positive. We will consider this complexity issue in the next later. Here, we only include the result for space like momenta. And then it means q_{\parallel}^2 being less than $-2\sqrt{M_n^2 q_0^2}$, where $M_n^2 = M^2 + 2neB$. Clearly, for $q_{\parallel}^2 < 0$, the saddle point locates at

$$x_0 = \frac{1}{2} - s \frac{\text{sign}(q_0)}{2} \sqrt{\frac{4E_k^2 q_0^2 - q_{\parallel}^2 q_0^2}{q_{\parallel}^4 - q_{\parallel}^2 q_0^2}}. \quad (33)$$

It leads $Tf^{(2)}(x_0; q_0 \rightarrow q_0 \pm i\varepsilon) = 2|q_3|^3/\sqrt{q_{\parallel}^4 - 4E_k^2 q_{\parallel}^2} \pm i \text{sign}(q_0)\varepsilon$ being always positive for $s = \pm 1$. And then the answer of Eq. (32) becomes pure imaginary. Using the notation of $\langle f \rangle = \int f dx d\hat{k}_3$, one has

$$\begin{aligned} \langle \eta q_{\parallel}^2 \Xi_0^T(v \pm i\varepsilon) \rangle &= \frac{\pm i}{8\pi^{\frac{3}{2}}} \sum_{s=\pm 1} \sum_{n=0}^{\infty} \int_{M_n^2}^{\frac{q_{\parallel}^4}{4q_0^2}} \frac{T^{\frac{1}{2}}(2eB)q_{\parallel}^2}{|q_3|^{\frac{1}{2}} (\Lambda - M_n^2)^{\frac{1}{2}}} \frac{\partial}{\partial q_{\parallel}^2} \frac{\text{Li}_0 \left[-\exp \left(\frac{sq_0}{2T} - \frac{\sqrt{q_{\parallel}^4 q_3^2 - 4\Lambda q_{\parallel}^2 q_3^2}}{2q_{\parallel}^2 T} \right) \right]}{(q_{\parallel}^4 - 4\Lambda q_{\parallel}^2)^{\frac{1}{4}}} d\Lambda \\ &\simeq \frac{\mp i}{8\pi^{\frac{3}{2}}} \sum_{n=0}^j \frac{(2eB)(q_{\parallel}^4 - 4q_0^2 M_n^2)^{\frac{1}{2}}}{3T^{\frac{1}{2}}|q_0|^{\frac{3}{2}}} \text{Li}_{-1} \left(-e^{-\frac{q_3^2}{2|q_0|^T}} \right) \end{aligned} \quad (34)$$

where $j = \lfloor q_{||}^4/(8eBq_0^2) - \hat{M}^2 \rfloor$.

In this paper, the requirement $\text{Re } \chi > 0$ of Eq. (31) is satisfied always. λ is non-positive when $q_{||}^2 > 4M_n^2$, which induces instability of exponential integration. Considering such case, we express the result by the Cauchy's principal value method,

$$\text{Im} \left[\int_0^\infty \tau^{\nu-1} \exp \left(\lambda \tau - \frac{\chi}{\tau} \right) d\tau \right] = \pi \left(\frac{\chi}{\lambda} \right)^{\frac{\nu}{2}} J_\nu \left(2\sqrt{\lambda\chi} \right), \quad (35)$$

where $J_\nu(z)$ is the Bessel functions of the first kind; $\text{Re } \lambda > 0$ and $\text{Re } \chi > 0$. Due to the formula of $J_\nu(e^{im\pi}z) = e^{im\nu\pi} J_\nu(z)$ for integer m , one gets $J_\nu(z)|_{p_0 \rightarrow p_0 + i\varepsilon} = -J_\nu(z)|_{p_0 \rightarrow p_0 - i\varepsilon}$ for half integer ν .

At $\nu = \frac{1}{2}$, one has $J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z$. Therefore, we obtain

$$\sum_{s=\pm 1} \sum_{l=1}^\infty l^{\frac{1}{2}} |y|^{-\frac{1}{4}} J_{\frac{1}{2}}(l\sqrt{-y}) \exp(sl\xi) \cos(l\pi) = \sqrt{\frac{2}{\pi}} \frac{-2 \sin \sqrt{-y}}{\sqrt{-y} (e^\xi + 2 \cos \sqrt{-y} + e^{-\xi})}, \quad (36)$$

which is exponentially suppressed in the low temperature limit for $|q_0| \gg T$. Hence, we will render only the results at high temperatures. Obviously, in the strong magnetic fields, the allowed momenta region $q_0^2 > q_{||}^2 > 2neB \gg T^2$ is extremely precluded at finite n . The lowest Landau level is left and contributes an imaginary part in the time like momenta, which is similar to the results at zero temperature. Meanwhile, for $\xi \ll 1$ and $\sqrt{-y} \ll 1$, one has $e^\xi + 2 \cos \sqrt{-y} + e^{-\xi} \simeq 4$ and gets

$$\langle (\eta q_{||}^2 - M^2) \Xi_0^T(v \pm i\varepsilon) \rangle \simeq \mp \frac{i(2eB)}{8\pi^2 T} \int_{\frac{1-\bar{x}}{2}}^{\frac{1+\bar{x}}{2}} dx \int_0^{\bar{k}_3} \left(\frac{q_{||}^2 \partial}{\partial q_{||}^2} + \frac{M^2 \partial}{\partial M^2} \right) \frac{\sin \sqrt{-y}}{\sqrt{-y}} dk_3 = \pm \frac{i(2eB) (q_{||}^2 - 4M^2)^2}{2^{11} \cdot \pi T^3 |q_{||}|}, \quad (37)$$

where $\bar{x} = \sqrt{1 - 4M^2/q_{||}^2}$ and $\bar{k}_3 = \sqrt{\eta q_{||}^2 - M^2}$.

It is easy to see we have two branch cuts along the real axis of q_0 . After extended it to the complex plane, one has $\text{Disc } \pi_i(q_0) = \pi_i(q_0 + i\varepsilon) - \pi_i(q_0 - i\varepsilon) = 2i \text{Im } \pi_i(q_0)$. In the strong magnetic field and chiral limit, we write down the discontinuities directly.

The first branch cut, $q_0^2 < q_3^2 + 2M_n^2 - 2\sqrt{M_n^4 + q_3^2 M_n^2} < q_3^2$, is developed at finite temperatures, which is corresponding to the process of $\gamma + \psi \rightleftharpoons \psi$. One has:

$$\begin{aligned} \text{Disc } \pi_1(q_0) &\simeq 2 \text{Im} \langle (M^2 + \eta q_{||}^2) (\Xi_0^T(v) + \Xi_0^T(v+1)) \\ &\quad + (2eB) (\Xi_2^T(v) + \Xi_2^T(v+1) + \Xi_4^T(v) + \Xi_4^T(v+1) + 2\Xi_0^T(v)) \rangle \\ &\simeq \left(\sum_{n=0}^j + \sum_{n=1}^j \right) \frac{-1}{4\pi^{\frac{3}{2}}} \frac{(2eB) (q_{||}^4 - 4q_0^2 M_n^2)^{\frac{1}{2}}}{T^{\frac{1}{2}} |q_0|^{\frac{3}{2}}} \text{Li}_{-1} \left(-e^{-\frac{q_3^2}{2|q_0|T}} \right) \\ \text{Disc } \pi_2(q_0) &\simeq \frac{q_{||}^2}{2eB} \text{Disc } \pi_1(q_0); \\ \text{Disc } \pi_3(q_0) &\simeq -\frac{q_{||}^2}{3q^2} \text{Disc } \pi_1(q_0); \\ \text{Disc } \pi_4(q_0) &= 0. \end{aligned} \quad (38)$$

Where $\text{Disc } \Pi^{\mu\nu}(q_0) = \sum_{i=1}^4 P_i^{\mu\nu} \text{Disc } \pi_i(q_0)$ as denoted before. Roughly speaking, the finite- n Landau level contribution is supposed to exponentially suppressed, e^{-n} , which is the underlying of the approximation of LLL. But, in a strictly manner, the n -th Landau levels present as $L_n^{(\alpha)} e^{-n}$. Indeed, the Laguerre polynomial was neglected improperly in lots of early works. For large n , the asymptotic behavior of $L_n^{(\alpha)}$ is limit to $n^{\frac{\alpha}{2}-\frac{1}{4}}$. Hence, $L_n^{(\alpha)} e^{-n}$ is characterized by a non-monotonic behavior of n when $\alpha \geq 1$. In other words, the LLL approximation will break down if the terms contained $L_n^{(1)}$ play an important role in the estimation.

The second branch cut, $q_0^2 > q_3^2 + 4M^2$, is due to the conventional process $\gamma \rightleftharpoons \psi + \psi$. In the chiral limit $M \rightarrow 0$, One has:

$$\begin{aligned} \text{Disc } \pi_1(q_0) &\simeq 2 \text{Im} \langle (\eta q_{||}^2) \Xi_0^T(v) - (2eB) (\Xi_2^T(v) + \Xi_4^T(v)) \rangle = \frac{(2eB)|q_{||}|}{2^{11} \cdot \pi T^3}; \\ \text{Disc } \pi_2(q_0) &= 0; \\ \text{Disc } \pi_3(q_0) &= -\frac{q_{||}^2}{q^2} \text{Disc } \pi_1(q_0); \\ \text{Disc } \pi_4(q_0) &= 0. \end{aligned} \quad (39)$$

VI. CONCLUSION AND DISCUSSIONS

In this work, we have completed the form of the photon polarization tensor at finite temperature in the media of a static and homogeneous magnetic field. In our process, the summing of Matsubara frequency was applied elegantly, since its concise formulation limit well and achieve easily in the approach of proper time representation. Meanwhile, it was argued that the summation of Matsubara frequency is not commuted with Landau level ones, such problem does not occur in our calculations. As expected, without a divergent environment included, the summation of Landau level is convergent. Before, one has to attend the cutoff of the Landau level via a test of numerical method. Unlike such cumbersome way, we truncate the Landau level in a systematic consideration since the dependence of Landau level is expressed in an obviously manner.

It is well known that the self-energies of gauge bosons take the same forms for both QED and QCD plasmas in the limit of long wavelength [41]. It takes place in the strong magnetic field limit, as well. In the space like momenta regime $q_3^2 \gg q_0^2$ and strong B -fields limit $(2eB) \gg q_0^2, q_3^2$, it is allowed us to define $(2eB) \sim \lambda T^2$ and $\lambda > 1$. The classification of the energy scale is similar to hard-loop action, where loop momenta $k \sim M_n$ for finite- n Landau levels while as the external momenta $q_3 \sim \lambda^{-\frac{1}{2}} T^{\frac{1}{2}} M_n^{\frac{1}{2}}$, $q_0 \sim \lambda^{-\frac{3}{2}} T$. Described by the result of Eq. (38), $\text{Disc } \pi_1$ is at the order of $\lambda^{\frac{7}{4}} \text{Li}_{-1}(-e^{-\lambda})$, which is not monotonically decreasing as λ increasing. Physically, such unique feature are essentially same as other gauge theories governed plasma systems. This large imaginary part only arises at finite temperatures, which is the so called Landau damping. It demonstrates the absorption of soft fields by hard plasma constituents. Such kind of processes gains the contribution not only from the lowest (soft) Landau level but also up to the finite- n (hard) levels. The general mathematic explanations have been analyzed as before. Eventually, we conclude that the LLL approximation is suit at zero temperature but not well described at the finite temperatures. In particular, the hard-loop approximation takes control for the magnetized plasma systems, whose constituent of typical momentum is much larger than the probing wave vector.

Our results have two-fold applications in the next future. One is that the Landau damping effect from the finite Landau levels has the potential to solve the inverse magnetic catalysis which observed in the lattice simulation. The second is that the modification of the photon production rate is a new effect which has to be considered for discussing photon v_2 puzzle in heavy ion collisions.

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